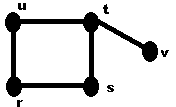
1. Let be an integer greater than , and let be a graph with vertices and edges. that is connected. That means that has a spanning tree, , by the theorem we proved in class. By the definition of a spanning subgraph . So, has vertices. Since is a tree, it has edges, by the theorem proved in class. This contradicts the assumption, because a spanning tree cannot have more edges than the graph it belongs to ⇒⇐. Therefore is not connected.
2. Let be a graph. Suppose has a unique path. So, is connected. that is not a tree. Since is connected and not a tree, has at least one cycle. Consider the graph below.

Note that the graph is connected with one cycle. However, there are paths, namely and so there is no unique path ⇒⇐. So, for to have a unique path, has to be connected and acyclical. Therefore is a tree.

1. Let be a tree with at least vertices and let . Assume is connected. that is not a leaf. That is . Since is a tree, it is connected, so . Thus . By the theorem proved in class, every edge is a cut edge in. So, when deleting , edges connected to are also deleted. Since every edge is a cut edge, then the deletion of causes to be broken into more components, so would not be connected ⇒⇐. Therefore is a leaf.
2. Let be a tree and let have vertices. So, . By the handshake theorem, the sum of the degrees is , or . To find the average degree, you divide the sum of the degrees by the number of vertices. So the average is , or , which is less than . Therefore the average degree of a vertex in is less than .
3. Let G be a forest with n vertices and c components. Consider that G has 1 component. That means that G is connected. Thus G is a tree. So, G has n – 1 edges. Now consider deleting one edge of G. That means G is no longer a tree, but a forest with two components, and G has n – 2 edges. With every edge deletion, another component is formed, because every edge is a cut edge. Therefore G has n – c edges.